Paper III Group:

Group:— Let G be a non-empty set and o is binary operation then the system (G, o) is called group iff the following postulates are satisfied:

(i) closure law:— For all $a, b \in G$ such that $aob \in G$.
i.e. G is closed under the binary operation o

(ii) Associative law:— For all $a, b, c \in G$ such that
$(aob)oc = ao(boc)$
i.e. associative law for the operation o holds.

(iii) Existence of ~~identity~~ inverse elements:— For every $a \in G$ there exists an element $b \in G$ such that $aob = boa = e$, e being the identity element
i.e. the inverse elements of all the elements of G exist. The inverse of a denoted by a^{-1}.

(iv) Existence of identity element:— For all $a \in G$ and there exists an element $e \in G$ such that $aoe = eoa = a$.
i.e the identity element exist in G.

Theorem:— Show that the identity element in a group is unique.
or
Show that a group has a unique neutral element.

Proof:— let suppose that a group (G, o) has two identity elements e and e', where $e \neq e'$.
When e is identity element then from the law of existence of identity element of a group, we have
$$aoe = eoa = a \text{ ——— } (1) \ \forall a \in G$$
And when e' is identity of G then
$$aoe' = e'oa = a \text{ ——— } (2) \ \forall a \in G$$
Here the equality of (1) and (2) show that
$e = e'$
This lead our contradiction that $e \neq e'$. Hence our supposition is wrong. So the identity element of a group is unique.

2. Theorem :- Show that every element of a group has a unique inverse.

or

Prove that the inverse element of an element in a group is unique.

Proof :- Let (G, o) be a group and a be any element of G. Suppose x and y are two inverses of a then. We have

$$aox = xoa = e \quad [\because x \text{ is inverse of } a] \text{ ——(1)}$$

$$aoy = yoa = e \quad [\because y \text{ is inverse of } a] \text{ ——(2)}$$

from above two relations, we have

$$aox = aoy = e$$

$$\Rightarrow xo(aox) = xo(aoy) \quad [\text{multiply left by } x]$$

$$\Rightarrow [xoa]ox = (xoa)oy \quad (\text{By associative law})$$

$$\Rightarrow eox = eoy \quad (\text{by } (1))$$

$$\Rightarrow x = y \quad [\because e \text{ is identity}]$$

This shows that any two inverse of 'a' must be identical. Hence every element of a group G is unique.

Proved